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LOCATIONAL EQUILIBRIUM OF TWO FACILITIES ON A TREE

by

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Université de Montréal
Centre de recherche sur les transports - Publication #696
May 1990

Dépôt légal - 2e trimestre 1990

ABSTRACT

This paper considers the location of two facilities on a tree. In particular, conditions for the existence of an equilibrium are derived and an efficient computational procedure is described for finding such an equilibrium, if one exists. Secondly, conditions for the so-called "first-entry paradox" are examined: in this, the first entrant to a market may not always have an advantage over his opponent as commonly thought. Relationships between the occurrence of this paradox and the existence of a Nash equilibrium are demonstrated.

Key words : locational equilibrium, Nash equilibrium, trees.

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RÉSUMÉ

Dans cet article, on considère la localisation de deux installations sur un arbre. En particulier, des conditions pour l'existence d'un équilibre y sont développées. On y décrit de plus une procédure efficace pour la détermination d'un tel équilibre lorsqu'il en existe un. En deuxième lieu, on étudie les conditions d'existence du "paradoxe du premier joueur", selon lequel le premier joueur ne possède pas nécessairement un avantage sur son adversaire. On établit finalement des relations entre l'existence de ce paradoxe et celle d'un équilibre de Nash.

Mots-clés : équilibre de localisation, équilibre de Nash, arbres.

1. INTRODUCTION

Hotelling's [14] fundamental paper concerning a spatial model with two facilities has influenced and inspired a large number of researchers interested in competitive location models. Hotelling's model was generalized as early as 1937 by Lerner and Singer [17] and later by Smithies [25]. Recent introductions to the field and related surveys are those by Gabszewicz and Thisse [10] and Eiselt and Laporte [5]; a taxonomy and research bibliography is found in Eiselt et al. [8].

Traditionally, prices and/or locations of the given facilities have been the decision variables in these models. The difficulties arising in models in which prices and locations are determined simultaneously have been demonstrated by a number of authors, e.g., d'Aspremont et al. [2]. Often, economists work with two-stage models where location is determined in the first stage and prices are found in the second stage. Such a two-stage model was originally developed by Hotelling; some newer references are Osborne and Pitchik [20], Capozza and Van Order [1] and Hamilton et al. [13]. On the other hand, geographers frequently consider models with parametric prices and discuss scenarios in more realistic spaces – e.g., Eaton and Lipsey [3], Osborne and Pitchik [19] and Fujita and Thisse [9]. Following the latter strand of research, we fix the price of the homogeneous good under consideration and use only the locations of the two facilities as variables. Both facilities are assumed to be independently operated and their objective is assumed to be the maximization of their respective sales.

Most traditional models locate facilities on a linear market, i.e. a line segment, although there are some notable exceptions, such as Okabe and Suzuki [18], ReVelle [24] and Eiselt and Laporte [6,7] who locate facilities in the plane, on networks and on trees, respectively. In this

study, both facilities locate on vertices of a tree. Also, demand originates only at vertices and is assumed to be totally inelastic with respect to distance or service level, i.e. customers will satisfy their demands no matter how far or unattractive the facility they buy from.

Formally, define a tree $T = (V,E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and edges e_{ij} for $v_i, v_j \in V$. Let $b_j > 0$ denote the demand at vertex v_i. Furthermore, $d_{ij} > 0$ denotes the distance between v_i and v_j; this is either the direct distance if $e_{ij} \in E$ or the sum of direct distances of all edges on the (unique) path from v_i to v_j if $e_{ij} \in E$. Let now two facilities A and B be given. With each facility we associate weights w_A and $w_B > 0$ that indicate the attractiveness of the facility. The specific interpretation of the weights depends on the example; in the case of shopping centers this could be floor space, for libraries it could be the number of books, or it might be the number of rooms if the facilities represent hotels. Suppose now that the attraction of a customer to a facility is measured by an attraction function. In particular, a customer at vertex v_i is attracted to facility A at v_A according to w_A/d_{iA}^r with $r \geq 1$. This attraction function is a generalization of the gravitational models used by Reilly [23], Huff [15] and Huff and Jenks [16]. A customer will now patronize the facility he is more attracted to. Clearly, for any given pair of locations of A and B, each customer at vertex v_i is either attracted more to A (in which case he will satisfy his entire demand b_i from A) or to B (he then buys only from B) or he is equally attracted to both facilities, in which case each of the two facilities will satisfy half that customer's demand. Moreover, if we allow both facilities to locate at the same vertex, say v_i, we can distinguish between the case where the larger facility A captures the entire market ("winner takes all") in contrast to the "proportional model", where customers at v_i will buy $[w_A/(w_A + w_B)]b_i$ units from facility A and the remaining $[w_B/(w_A + w_B)]b_i$ units from B.

Consequently, for any fixed pair of facility locations, each facility could determine the vertices whose customers it serves (or captures). Formally, define $V_{ij}(A)$ and $V_{ij}(B)$ as the sets of vertices captured by A and B, respectively, given that A locates at vertex v_i and B locates at vertex v_j. Vertices buying from both facilities are included in both sets. $V_{ij}(A)$ and $V_{ij}(B)$ are called market (or catchment) areas; the analog in R^m would be the Voronoi set of a given point.

Naturally, each facility may now locate in order to maximize its sales. Suppose now that it were possible to relocate sequentially and repeatedly. The concept of sequential relocation coupled with facilities that optimize on the basis of the current situation has frequently been challenged. Indeed, if say, facility A were able to relocate now, why would it do so on the basis of the current situation knowing that facility B will relocate next and quite possibly negate the benefit that A derives from its planning? In short, why not anticipate B's move and react accordingly? Such planning with foresight has successfully been applied in the case of linear markets, e.g., Prescott and Visscher [22]. Suppose now that it were known to all players that they will relocate in a specific sequence with unknown relocation speeds. As it may not be desired to anticipate an opponent's move some time in the (possibly distant) future, a possible short-term objective is the maximisation of sales within the time interval where no other facility relocates.

Finally, define a pure strategy locational Nash equilibrium as a pair of locations (v_i^*, v_j^*), so that neither facility may gain by relocating unilaterally. A formal characterization of such an equilibrium in the context of two-person games is provided below in Definition 2.

The remainder of this paper is organized as follows. In Section 2, we determine equilibrium conditions for two facilities on a tree. In Section 3, we analyze the equilibrium problem as a two-person constant-sum game. We then study, in Section 4, the so-called "First Entry Paradox" which states that the first facility to enter in a sequential location procedure does not necessarily have an advantage. The conclusion follows in Section 5.

2. EQUILIBRIUM CONDITIONS FOR TWO FACILITIES ON A TREE

In this section, we investigate existence conditions for the equilibrium of two facilities on a tree. To facilitate the discussion, we use the following notation. Let v_q denote any vertex in the tree T. Deletion of v_q and of all its indicent edges decomposes the tree into subtrees T_1^q, T_2^q, ..., $T_{d(q)}^q$, where d(q) denotes the degree of v_q. Suppose now that the subtrees are ordered so that $b(T_1^q) \geq b(T_2^q) \geq \ldots \geq b(T_{d(q)}^q)$ and let the demand of any subtree T_k^q be $b(T_k^q) = \sum_{v_i \in T_k^q} b_i$; the demand of T is defined in a similar fashion.

2.1 Facilities with equal weights

Assume first that the two facilities that are to be located on the tree may share a vertex. Let one of the facilities, say A, be located at vertex v_q. Similar to the result on the linear market, facility B will either locate adjacent to v_q or also at v_q. If B locates adjacent to v_q, it will do so in T_1^q, as B captures the demand of the subtree in which it is located and, by definition, T_1^q is the largest such subtree. If v_q happens to be a median of T, then $b(T_1^q) < b(T)/2$, as shown by Goldman and Witzgall [12]. If A and B both locate at a median, each facility captures b(T)/2. As the maximum demand either facility can get by moving out of v_q is $b(T_1^q)$, there is no incentive to relocate and hence a Nash equilibrium has been

reached. An example for such a case is shown in Figure 1 where $v_q = v_6$ is the unique median; here, and in similar figures, the numbers next to the vertices indicate their respective weights.

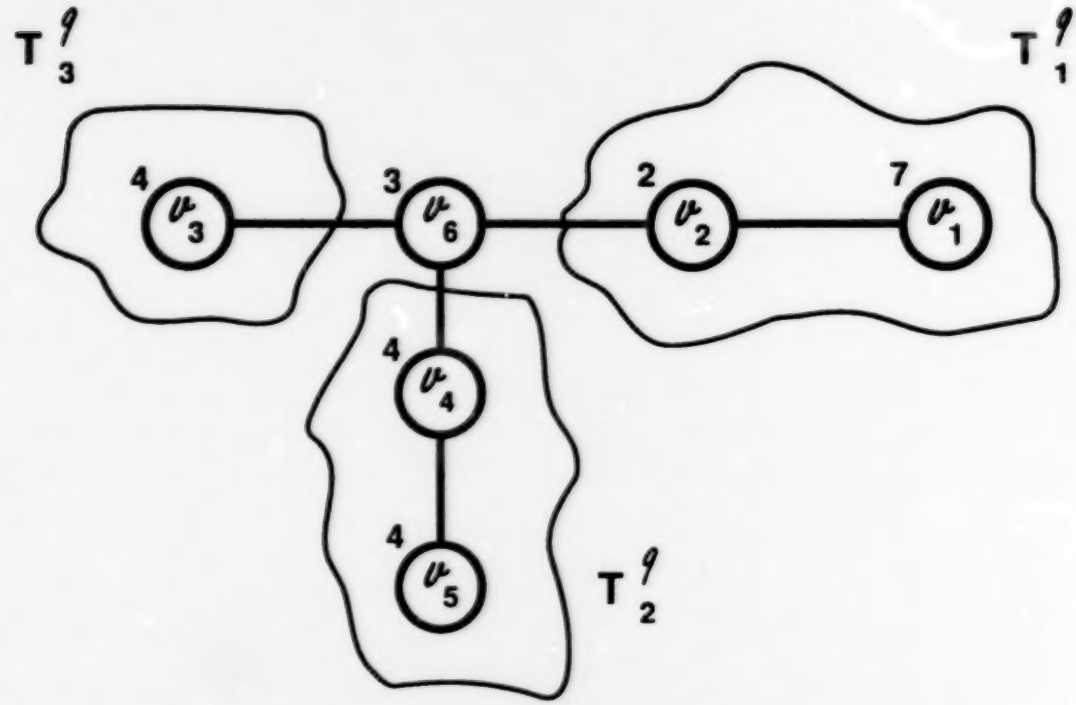


Figure 1: Nash equilibrium for two facilities with equal weights and locating at the unique median of a tree

The case of locating two equally weighted facilities at distinct vertices of a tree is similar. Suppose again that facility A has already located at the median v_q and, as above, the best B can do is locate next to v_q in T_1^q. Facility B then satisfies a demand of $b(T_1^q)$ whereas A captures $b(T)-b(T_1^q) > b(T_1^q)$. Once more, neither facility can improve its position by relocating unilaterally and thus a Nash equilibrium has again been reached.

2.2 Facilities with unequal weights

Consider now the case where the two facilities have unequal weights. As opposed to the equally weighted case where both facilities are drawn to each other and their behavior is totally symmetric, we have to distinguish between the large and the small facility. For simplicity, consider first the case of unequally weighted facilities on a linear market. Eiselt [4] has shown that the large facility will always locate arbitrarily close to the small facility thus capturing almost the entire market. On the other hand, the smaller facility will try to locate at a certain distance away from its larger competitor. Formally, let the market extend from 0 to 1, let $w_A > w_B$ and assume that facility A is located at point x_A. Without loss of generality assume $x_A > \frac{1}{2}$. Then facility B will locate at some point x_B so that the left end of its market area $V_{AB}(B)$ coincides with the left end of the market at "0". In general, for any given pair of locations of the facilities at x_A and x_B, $V_{AB}(B)$ is a non-symmetric interval around x_B, with the shorter end facing A whereas $V_{AB}(A)$ is the complement of $V_{AB}(B)$. Usually $V_{AB}(A)$ is disconnected.

A similar argument is applicable to the location of two facilities at the vertices of a tree. Here also, A will always locate at the same vertex as B or adjacent to B, as there is nothing to gain by moving farther away. On the other hand, B will move away from A but, based on the discrete structure of the market, not necessarily so far that the end of its market coincides with a leaf of the tree. This implies

Lemma 1: If an equilibrium exists, then A and B are located at adjacent vertices.

Figure 2 shows such a situation where a locational equilibrium exists. Here, $w_A = 3$, $w_B = 1$ and $d_{ij} = 1$ for all $e_{ij} \in E$. Note that wherever B

locates, A will locate next to it and B will then capture only the demand of the vertex where it is located. Thus, if one of the facilities locates at v_3 and the other at v_5, neither has an incentive to move out of this arrangement.

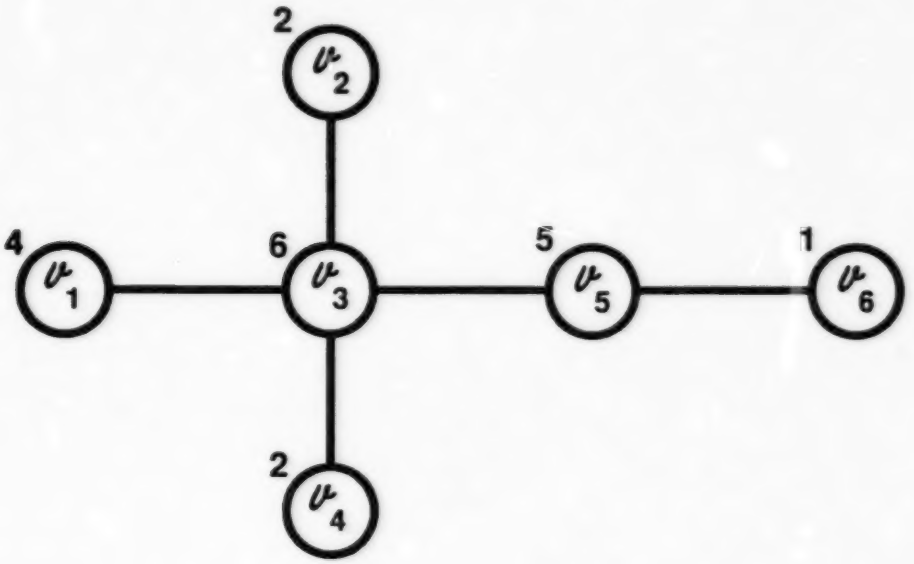


Figure 2: Nash equilibrium for two facilities with unequal weights

3. ANALYSIS OF THE EQUILIBRIUM PROBLEM AS A TWO-PERSON CONSTANT-SUM GAME

In order to further analyze the situation, we will model the problem as a two-person constant-sum game. For that purpose, construct a payoff matrix P, so that the rows correspond to potential locations of facility B, the columns correspond to potential locations of facility A and $p_{ij} = b_{ij}(B)$ denotes the payoff of B if B locates at v_i and A locates at v_j. Note that p_{ii} is not defined in contexts where A and B must locate at distinct vertices. Also note that P is not symmetric (however, if $w_A = w_B$, then $p_{ij} = b(T)-p_{ji}$). First we will outline a procedure for the determination of P

and then show how to find equilibria (if they exist), provided that the matrix P is available.

For each given pair of locations of A and B, $V_{ij}(B)$ can be computed by determining the attractions of A and B at each customer location and finding the maximum. For each vertex, this is accomplished in O(1) time. Since this task has to be repeated for all vertices, each element of the matrix P can be found in O(n) time. As there are $O(n^2)$ elements, such a procedure requires $O(n^3)$ time. It is summarized in the following steps:

Procedure payoff matrix:

Begin

$p_{ij}:=0, \quad \forall i,j; \; V_{ij}(B):=\emptyset, \quad \forall v_i, v_j \in V;$

for i:=1 **to** n **do**

for j:=1 **to** n **do**

for k:=1 **to** n **do**

begin

if $w_A/d_{ik}^r < w_B/d_{jk}^r$ **then** set $p_{ij}:=p_{ij}+b_k$ and $V_{ij}(B):=V_{ij}(B) \cup \{v_k\}$;

if $w_A/d_{ik}^r = w_B/d_{jk}^r$ **then** set $p_{ij}:=p_{ij}+b_k/2$ and $V_{ij}(B):=V_{ij}(B) \cup \{v_k\}$

end

end

Knowledge of the entire matrix P is however not necessary. In order to find an equilibrium, Lemma 1 can be invoked to limit the number of required computations. As facilities will be adjacent at equilibrium, two possible

location configurations must be examined for each edge e_{ij}: A locates at v_i and B at v_j and vice versa. As there are n-1 edges in T, 2(n-1) pairs of locations have to be investigated leading to an $O(n^2)$ algorithm. For this, it suffices to relabel every edge e_{ij} by $e_{i(\ell),j(\ell)}$, where $\ell=1, \ldots, n-1$. Then in the above procedure, the first two "do" loops are replaced by "**for** ℓ:=1 **to** n-1 **do**" and the instruction "set i:=i(ℓ) and j:=j(ℓ)" is inserted before the first "if" statement.

Suppose now that a payoff matrix $P = (p_{ij})$ is available. Then we can formally define a locational Nash equilibrium as follows.

Definition 2: An element p_{i*j*} is a Nash equilibrium if $p_{i*j*} \leq p_{kj*}$ for all k and $p_{i*j*} \leq p_{i*\ell}$ for all ℓ. Alternatively, if $r_i = \min_j \{p_{ij}\}$ for all i denotes the row minima and $c_j = \max_i \{p_{ij}\}$ for all j are the column maxima, then p_{i*j*} is a Nash equilibrium if $r_{i*} = c_{j*}$.

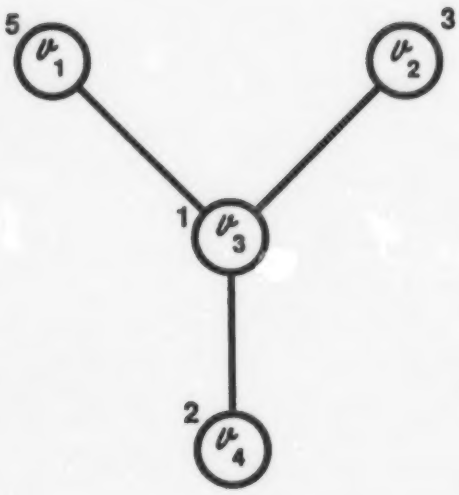
Similarly, a saddle point is defined as follows.

Definition 3: Let r_i and c_j denote the row minima and column maxima of matrix P, respectively. Define $r_{i'} = \max_i \{r_i\}$ and $c_{j'} = \min_j \{c_j\}$. Then an element $p_{i'j'}$ is a saddle point in P.

These definitions imply

Lemma 4: If a pair of locations (v_i, v_j) defines a saddle point, then it is also a Nash locational equilibrium.

The converse of Lemma 4 is however not true as shown in the example displayed in Figure 3, where $w_A = 3$, $w_B = 2$, $d_{ij} = 1$ for all $e_{ij} \in E$. Clearly, as $r_{i'} = 6 > 5 = c_{j'}$, P has no saddle point but the circled elements in the payoff matrix indicate locational Nash equilibria.



P:

	v_1	v_2	v_3	v_4	r_i
v_1	-	5	(5)	5	5
v_2	3	-	3	3	3
v_3	(6)	8	-	9	6
v_4	2	2	2	-	2
c_j	6	8	5	9	

Figure 3: Existence of a Nash equilibrium when there is no saddle point

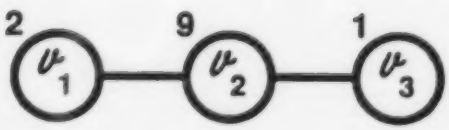
3.1 Facilities must locate at distinct vertices

Consider now the case where the facilities have to locate at distinct vertices. In such a case, an equilibrium may or may not exist. Figure 3 provides an example for the case in which equilibria do exist; (v_3, v_1) and (v_1, v_3) are such equilibria. On the other hand, an example for the case without an equilibrium is shown in Figure 4. Here, $w_A = 3$, $w_B = 1$, $b_i = 1$ for all $v_i \in V$ and $d_{ij} = 1$ for all $e_{ij} \in E$. In the associated payoff matrix, $r_i = 1$, $i = 1, \ldots, 11$ and $c = (c_j) = (5, 9/2, 4, 4, 3, 5/2, 3, 4, 4, 9/2, 5)$. As $r_i < c_j$ for all p_{ij}, there is no equilibrium.

Figure 4: Tree in which no equilibrium exists

3.2 Facilities may share the same vertex

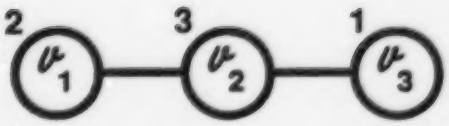
Let us now examine the model where it is possible for both facilities to locate at the same vertex. As indicated above, we will distinguish between the "proportional" and the "winner takes all" models. First consider the proportional model. Here, an equilibrium may or may not exist. Both possibilities are shown in Figures 5 and 6. In both examples, $w_A = 3$, $w_A = 1$, $d_{ij} = 1$ for all $e_{ij} \in E$. Whereas the model in Figure 5 has exactly one equilibrium with both facilities locating at v_2, the graph in Figure 6 shows no equilibrium.



P:

	v_1	v_2	v_3	p_i
v_1	1/2	2	2	1/2
v_2	9	9/4	9	9/4
v_3	1	1	1/4	1/4
c_j	9	9/4	9	

Figure 5: Tree with one equilibrium



P:

	v_1	v_2	v_3	r_i
v_1	1/2	2	2	1/2
v_2	3	3/4	3	3/4
v_3	1	1	1/4	1/4
c_j	3	2	3	

Figure 6: Tree with no equilibrium

In our last model, both facilities may locate at the same vertex and if they do, the larger of the two facilities will capture the entire market. Assume now that there are at least two facilities with positive demands, say v_i and v_k. Then $p_{ij} > 0$ and $p_{ii} = 0$ for all $i \neq j$; $p_{kj} > 0$ and $p_{kk} = 0$ for all $k \neq j$. This implies that there exists at least one positive element in each column of P. Then by construction, $r_i = 0$ for all i and $c_j > 0$ for all j and hence no equilibrium can exist in this case.

The above results are summarized in Table 1.

Table 1: Existence of equilibria

Facility weights		Facilities may locate at the same vertex	Facilities must locate at different vertices
Equal		Yes, always (at median)	Yes, always (at median and adjacent vertex)
Unequal	Proportional model	Sometimes, but not always	Sometimes, but not always
	Winner takes all	Never	

4. THE FIRST ENTRY PARADOX

The concept of the first entry paradox has recently been put forward by Ghosh and Buchanan [11]. The basic idea behind the principle is that the first facility to enter the market in a sequential location procedure has an advantage as it can locate at a strategic site. Sometimes, however, facilities locating later in the process may have "the last word" and capture a larger share of the market than the facilities located earlier. Such a situation is referred to as the first entry paradox. Note that the paradox assumes a completely symmetric situation, i.e. both facilities have access to the same information, resources, etc. Clearly, this assumption is not satisfied in a model where facility A has an obvious advantage by virtue of its larger weight and thus larger attraction. This necessitates a slightly different formulation fo the paradox. Define $V'_{AB}(A)$ as the market area captured by facility A if A locates before B and let $V'_{BA}(A)$ be A's market share if B locates first; the definitions of $V'_{AB}(B)$ and $V'_{BA}(B)$ are similar. Furthermore, define $b'_{AB}(A)$, $b'_{BA}(A)$, $b'_{AB}(B)$, and $b'_{BA}(B)$ as the demands associated with the respective market areas. Then we can write

Definition 5: The first entry paradox is said to exist if $b'_{AB}(A) < b'_{BA}(A)$ [or equivalently, if $b'_{BA}(B) < b'_{AB}(B)$].

Suppose now that the two facilities locate sequentially. In anticipation of their opponent's counterstrategy, the first facility to locate will employ a maximin criterion. Thus, if facility B were to locate first, it would determine $r_{i*} = \max\{r_i\}$. Then facility A determines its best counterstrategy given by the smallest element in row i* which is r_{i*}; thus $V'_{AB}(B) = r_{i*}$. Similarly, if A locates first it will choose column j* so that $c_{j*} = \min_j\{c_j\}$. Subsequently, B optimizes resulting in a payoff of c_{j*} for B, i.e. $V'_{AB}(B) = c_{j*}$. Hence, we can write

Lemma 6: The first entry paradox occurs if and only if $r_{i*} < c_{j*}$.

For the following discussion it is useful to restate one of the fundamental theorems in game theory (see, for example, Owen [21]).

Lemma 7: For any given payoff matrix $P = (p_{ij})$,

$$\max_i \min_j \{p_{ij}\} \leq \min_j \max_i \{p_{ij}\}.$$

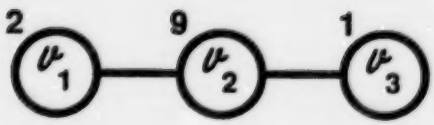
As above, we will distinguish between two cases.

4.1 Facilities may share the same vertex

Lemma 7 holds as long as all elements of P are defined, i.e. both facilities are allowed to locate at the same vertex. In this case, $\max_i \{r_i\} \leq \min_j \{c_j\}$, meaning that the paradox always exists, except when the equality holds. This is, however, precisely the case when the game has a saddle point. Thus we can state

Lemma 8 : If both facilities may locate at the same vertex, then the first entry paradox always occurs, except when the game has a saddle point.

Figures 5 and 6 illustrate Lemma 8. The example in Figure 5 has a saddle point and hence the paradox does not occur. On the other hand, for the graph in Figure 6, $r_{i*}=3/4 < 3=c_{j*}$ and thus, the paradox exists. Finally, Figure 7 shows an example of the "winner takes all" model. Here $w_A=3$, $w_B=2$, $d_{ij}=1$ for all $e_{ij} \in E$. Whenever B locates first, his payoff will be zero whereas if A locates first, B will be able to capture a demand of at least 2.



P:

	v_1	v_2	v_3	μ_i
v_1	0	2	2	0
v_2	9	0	9	0
v_3	1	1	0	0
c_j	9	2	9	

Figure 7: First entry paradox in a "winner takes all" model

4.2 Facilities must locate at distinct vertices

Consider now the model where the facilities have to locate at distinct vertices and where Lemma 6 does not apply. It is easy to show that the first entry paradox never occurs for equally weighted facilities. As has been shown elsewhere (see Eiselt and Laporte [7]), the first facility to locate, say A, would do so at a median of the tree, say v_q. Thus, $b'_{AB}(B) = b(T_1^q)$ and $b'_{AB}(A) = b(T) - b(T_1^q)$. By construction, $b(T_1^q) < b(T)/2$, so that $b'_{AB(A)} > b(T)/2 > b'_{AB}(B)$. Similarly, it can be shown that $b'_{BA}(B) > b'_{BA}(A)$ and as there is total symmetry in this case, $b'_{AB}(A) = b'_{BA}(B)$ and $b'_{AB}(3) = b'_{BA}(A)$, so that $b'_{BA}(B) = b'_{AB}(A) > b'_{AB}(B)$ which contradicts the condition for the first entry paradox.

In the case of facilities with unequal weights, the paradox may or may not occur. Examples for both cases are provided by the graphs in Figure 3 and 4 above. In the tree in Figure 3, B can obtain sales of 6 if it locates first but it is guaranteed sales of only 5 units if A is the first

facility to locate; here the paradox does not occur. In Figure 4, facility B will be guaranteed one demand unit if B moves first whereas it will capture at least 5/2 units if A moves first, indicating that the paradox does occur.

The above results are summarized in Table 2.

Table 2: Occurrence of the first entry paradox

<table>
<tr><th colspan="2">Facility weights</th><th>Facilities may locate at the same vertex</th><th>Facilities must locate at different vertices</th></tr>
<tr><td colspan="2">Equal</td><td rowspan="3">If and only if $\max_i\{r_i\} < \min_j\{c_j\}$</td><td>Never</td></tr>
<tr><td rowspan="2">Unequal</td><td>Proportional model</td><td rowspan="2">Sometimes, but not always</td></tr>
<tr><td>Winner takes all</td></tr>
</table>

5. CONCLUSION

In this paper, we have examined the existence of equilibria in a competitive duopoly model on a tree. Furthermore, conditions for the occurrence of the first entry paradox were developed. Future research could investigate equilibria and paradox situations in general graphs as well as in trees with more than two facilities.

ACKNOWLEDGMENT

This research was in part supported by the Natural Sciences and Engineering Research Council of Canada under grants OGP0009160 and OGPIN020. This support is gratefully acknowledged.

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